HEAT TRANSFER IN A DISPERSIVE MEDIUM BOUNDED BY NONBLACK SURFACES

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The problem of radiative exchange in a medium bounded by gray walls is reduced to a similar problem with black walls. The influence of a deviation from Lambert's law on radiation transmitted through a layer is examined.

§1. Radiation intensity in an absorbing and dispersive medium is described by the transfer equation

$$\vec{\Omega}\vec{\nabla} I + k I = \frac{k_s}{4\pi}\int P(\omega, \omega') I(\mathbf{r}, \omega') d\omega' + j. \tag{1}$$

This equation is linear in regard to intensity. If the volume is bounded by nonblack surfaces, the boundary conditions are written in the form

$$I^-(\Gamma, \omega) = I_0(\Gamma, \omega) + \\ + \frac{1}{2\pi}\int r(\Gamma, \omega') P'(\omega, \omega') I^+(\Gamma, \omega') d\omega'. \tag{2}$$

Integration in (2) is performed over half the solid angle. The problem of integrating Eq. (1) with boundary conditions (2) in the general case is highly complicated. It simplifies appreciably in the case in which Lambert's law is valid, i.e., where it is assumed that both the reflected and self-radiation of the surfaces are isotropic. We subdivide the surface Γ into Γ_k regions, in each of which the reflection factor and the intensity of the incident and self-radiation are constant. In this case

$$I^-(\Gamma_k) = I_0(\Gamma_k) + \frac{1}{2\pi}\int r_k(\omega') I^+(\Gamma_k, \omega') d\omega'. \tag{3}$$

Let us assume that we know the solution of Eq. (1) in the case of black walls, with the boundary conditions

$$I^-(\Gamma_k) = 1, \quad I^-(\Gamma_{i \neq k}) = 0$$

and a zero source function, and that we also know the solution with a given source function and zero boundary conditions

$$I^-(\Gamma_k) = 0.$$

We denote these solutions by $I'_k(\mathbf{r}, \omega)$ and $I'_j(\mathbf{r}, \omega)$, respectively. In view of the linearity of the transfer equation, the general solution for nonblack surfaces has the form

$$I(\mathbf{r}, \omega) = \sum_k \left[I_0(\Gamma_k) + I_s(\Gamma_k)\right] I'_k(\mathbf{r}, \omega) + I'_j(\mathbf{r}, \omega). \tag{4}$$

By substituting these expressions into Eq. (3), we get

$$I_s(\Gamma_k) = \sum_i \left[I_0(\Gamma_i) + I_s(\Gamma_i)\right] a_{ki} + b_k, \tag{5}$$

where

$$a_{ki} = \frac{1}{2\pi}\int r_k(\omega') I'^{+}_i(\Gamma_k, \omega') d\omega';$$

$$b_k = \frac{1}{2\pi}\int r_k(\omega') I'_j(\Gamma_k, \omega') d\omega'. \tag{5'}$$

Thus, knowing the solution for black surfaces, and having solved the system of linear equations (5), we get the intensity of the reflected radiation, while from formula (4) we obtain the total radiation. One frequently encounters the case of two surfaces (a plane layer, concentric spheres, coaxial cylinders). Here, one readily gets

$$I(\mathbf{r}, \omega) = \\ = \frac{(1-a_{22})(I_0(\Gamma_1)+b_1)+a_{12}(I_0(\Gamma_2)+b_2)}{(1-a_{11})(1-a_{22})-a_{12}a_{21}} I'_1(\mathbf{r}, \omega) + \\ + \frac{(1-a_{11})(I_0(\Gamma_2)+b_2)+a_{21}(I_0(\Gamma_1)+b_1)}{(1-a_{11})(1-a_{22})-a_{12}a_{21}} I'(\mathbf{r}, \omega) + \\ + I'_j(\mathbf{r}, \omega). \tag{6}$$

Assuming that the reflection factor is independent of the angle of incidence, for a plane isothermal layer it is not difficult to obtain from (6) an expression for the energy of the radiation incident on the first surface:

$$\varepsilon_1^+ = \left\{(\varepsilon_{01} R + \sigma T^4 \varepsilon_{\mathrm{rad}} + \varepsilon_{02}\varepsilon_{\mathrm{tr}}) + r_2(\varepsilon_{\mathrm{tr}} - R) \times \\ \times [\sigma T^4 \varepsilon_{\mathrm{rad}} + \varepsilon_{01}(\varepsilon_{\mathrm{tr}} + R)]\right\} \times \\ \times \left\{(1-r_1 R)(1-r_2 R) - r_1 r_2 \varepsilon^2_{\mathrm{tr}}\right\}^{-1}. \tag{7}$$

Obviously $\varepsilon_{\mathrm{rad}} + \varepsilon_{\mathrm{tr}} + R = 1$.

By interchanging subscripts $1 \rightleftarrows 2$, one obtains the energy of the radiation incident on the second surface.

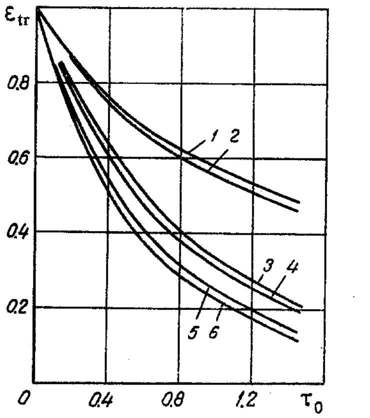
§2. If r_k is independent of the angle of incidence, one can use a somewhat different approach. It is similar to a method proposed by Vlasov for calculating the heat transfer between surfaces separated by a transparent medium [1]. Examine, for example, the equilibrium heat transfer between two closed surfaces exhibiting the properties mentioned above. Enclosed between these surfaces is an absorbing and dissipative medium with an arbitrary scattering characteristic. In the case of equilibrium, the energy released by the first surface is equal to the energy imparted to the second surface, i.e.,

$$\varepsilon = \varepsilon_1^- - \varepsilon_1^+ = \varepsilon_2^+ - \varepsilon_2^-.$$

We express the energy of the effective radiation in terms of intrinsic energy and the energy of the radiation incident on the surface:

$$\varepsilon_1^- = \varepsilon_{01} + r_1 \varepsilon_1^+, \quad \varepsilon_2^- = \varepsilon_{02} + r_2 \varepsilon_2^+.$$

Let us introduce the function G_{12}, which is equal to the portion of the energy emitted from the first surface



Energy of the passing radiation plotted vs. the radiation incident on the layer: 1) $\gamma = 1$, $I_0 = 1 + 3\mu/2$; 2) $\gamma = 1$, $I_0 = 2$; 3) $\gamma = 0.5$, $I_0 = 1 + 3\mu/2$; 4) $\gamma = 0.5$, $I_0 = 2$; 5) $\gamma = 0$, $I_0 = 1 + 3\mu/2$; 6) $\gamma = 0$, $I_0 = 2$.

that is absorbed by the second surface, and also the function G_{21}, which is equal to the portion of energy (emitted from the second surface) that is absorbed by the first surface, provided that both surfaces are black. Then obviously

$$\varepsilon_2^+ = \varepsilon_1^- G_{12} + (1 - G_{21}) \varepsilon_2^-,$$

since part of the energy $1 - G_{21}$ is reflected from the medium back to the second surface.

Having solved the obtained system of equations with respect to ε, we have

$$\varepsilon = \frac{G_{12} S_1 \sigma T_1^4 - G_{21} S_2 \sigma T_2^4}{\frac{r_1}{1-r_1} G_{12} + \frac{r_2}{1-r_2} G_{21} + 1},$$

where $\varepsilon_{01} = S_1 \sigma T_1^4 (1 - r_1)$, $\varepsilon_{02} = S_2 \sigma T_2^4 (1 - r_2)$ and S_1, S_2 are the areas of the first and second surfaces, respectively. Since, $\varepsilon = 0$ for $T_1 = T_2$, it follows that

$$G_{12} S_1 = G_{21} S_2,$$

i.e., a reciprocity relation is obtained. Taking this into account, we obtain

$$\varepsilon = G_{12} S_1 \sigma \frac{T_1^4 - T_2^4}{1 + \frac{r_1}{1-r_1} G_{12} + \frac{r_2}{1-r_2} G_{21}}. \quad (8)$$

For a plane layer $G_{12} = G_{21} \equiv G$; $\varepsilon / S_1 = Q$ (the energy flux). The function $G(\tau_0)$ (τ_0 is the optical thickness of the layer) has been numerically calculated in [2] for a layer free of scattering or having a spherical scattering characteristic. For a layer with a scattering characteristic of a simple elongated shape $P(\theta) = 1 + x \cos \theta$, we have calculated the function G for a plane layer in the first approximation by the method of spherical harmonics [3]

$$G(\tau_0) = \frac{1}{1 + \frac{3}{4} \tau_0 \left(1 - \gamma \frac{x}{3}\right)}, \quad (9)$$

where $\gamma = k_s / k$.

We have made use of Marshak's boundary conditions [3]. For two concentric spheres and two infinite coaxial circular cylinders separated by an absorbing and dispersive medium, the functions G_{12} also can be readily calculated in first approximation by the method of spherical harmonics. Assuming a spherical scattering characteristic, and making use of Marshak's boundary conditions, one obtains for concentric spheres,

$$G_{21} = \frac{1}{R_2^2} \times$$
$$\times \frac{1}{\frac{1}{2}\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right) + \frac{3}{4} k \left(\frac{1}{R_1} - \frac{1}{R_2}\right)}, \quad (10)$$

where R_1 is the radius of the inner sphere, and R_2 is the radius of the outer sphere. Using the same assumptions, for two coaxial cylinders, we obtain

$$G_{21} = \frac{1}{R_2'} \frac{1}{\frac{1}{2}\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) + \frac{3}{4} k \ln \frac{R_2'}{R_1'}}, \quad (11)$$

where R_1' and R_2' are the radii of the inner and outer cylinders, respectively.

§3. It is of interest to determine, if only qualitatively, the error introduced by assuming validity of Lambert's law. To this end, we have calculated in first approximation, by Ivon's method [3], the energy of the radiation transmitted through an absorbing and dispersive layer with a spherical scattering characteristic. The radiation intensity was approximated by the expression

$$I(\mu, \tau) = \varphi_0(\tau) + \mu \varphi_1(\tau) \quad (\mu > 0),$$
$$I(\mu, \tau) = \varphi_0'(\tau) + \mu \varphi_1'(\tau) \quad (\mu < 0),$$

where μ is the cosine of the angle formed between the axis normal to the planes and the direction of the radiation.

The intensity of the radiation emitted from the surface $\tau = 0$ was taken in the form $I_0 = 1 + (3/2)\mu$ and, for comparison, in the form $I_0 = 2$. The surfaces $\tau = 0$ and $\tau = \tau_0$ were postulated to be purely absorbing surfaces. In the figure, the curves showing the energy transmitted through the layer as a function of the optical thickness of the layer τ_0 are plotted for various values of γ. It can be seen from the figure that for moderately thick layers, a pronounced difference between the nonspherical and spherical radiation intensities has only a slight effect on the energy transmitted through the layer. For large optical thicknesses, the error in transmitted energy that arises in the substitution of spherical for nonspherical surface radiation does not increase ad infinitum, but rather tends to a certain constant value. Thus, for a nonscattering layer, in the case under consideration, the error approaches 0.2 as $\tau_0 \to \infty$.

NOTATION

Ω is the unit vector of radiation; k is the attenuation factor of the medium; k_s is the scattering coefficient of the medium; P is the scattering characteristic; ω is the solid angle of beam orientation; j is the source function, $I^-(\Gamma, \omega)$ is the effective surface radiation intensity; $I_0(\Gamma, \omega)$ is the surface self-radiation intensity; $I^+(\Gamma, \omega)$ is the intensity of radiation incident on the surface; $r(\Gamma, \omega)$ is the surface reflection factor; P' is the reflection characteristic; $I_s(\Gamma_k)$ is the reflected radiation intensity near the surface of Γ_k; ε_1^+ is the energy incident on the first surface; ε_{01} is the proper energy of radiation from first surface; ε_{02} is the proper energy of radiation from second surface; T is the temperature of the medium; σ is the Stefan-Boltzmann constant; ε_{rad} is the dimensionless energy of radiation of the layer; ε_{tr} is the portion of the energy transmitted through the layer; ε_1^- is the effective radiation energy of the first surface; ε_2^- is the effective radiation energy of second surface; τ is the optical thickness; **r** is the radius vector of a point; R is the reflection factor of the layer.

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